Mathematics textbooks: Messages to students and teachers

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The set textbook has a strong influence on what takes place in the mathematics classroom, especially at the secondary level. This paper reports on the preliminary stage of a project aimed at developing more effective textbook presentations. In this phase of the project, textbooks are being examined to identify the types of messages about mathematics and its teaching and learning inherent in their presentations and to identify the sources of the underlying messages.

Introduction

Textbooks are widely used in mathematics classrooms in Australia and many other parts of the world at all levels of schooling. Fauvel (1991) described the textbook as one part of a "book / pupil / teacher triangle" (p. 111) and noted that the approach in the textbook is a constraint on what the teacher can do in the classroom. In a survey of textbook usage across a range of subject areas in Australia, Morris (1989) observed that it was often the teacher who read and learnt from the textbook and not the students. Shield (1991) reported on a study of seven mathematics teachers and their uses of the set textbook with their year 8 classes. In this study it was the teachers and not the students who read the textbooks and used them in their pedagogic decision making. Students use was mostly restricted to doing the extensive sets of exercises which accompanied each section of the book, with occasional reference to a worked example. It was also noted that the way the textbooks were written did not provide encouragement for students to seek information.

There have been many calls for an increase in the range of strategies used in the teaching and learning of mathematics. Examples include the documents *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989), *Everybody Counts* (National Research Council, 1989) and *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991). These reports generally call for a broadening of teaching approaches, a greater emphasis on understanding, and a stress on the usefulness of mathematics through applications and problem solving. However, the teaching of mathematics in schools has generally been at variance with the ideas being developed in these documents. For example, Baroody and Ginsburg (1990) describe the "tell-show-do" approach in which the teacher firstly tells the students what they need to know, follows this by showing an example, then the students do a number of similar examples themselves.

The study reported in the present paper is the preliminary phase of a project to develop more effective models for the presentation of print and eventually on-line mathematical resources that are more closely aligned with recent reports and curriculum documents and with research into the ways students learn mathematics with understanding. Specifically, this part of the study looked at the presentations in some current school mathematics textbooks at the lower secondary level with a view to identifying the inherent messages about mathematics and its teaching and learning and the sources of these messages. The discrepancies with the thrust of current curriculum documents was also examined. While it is yet to be determined just how textbook mathematics can be presented in order to take full account of the current directions of mathematics education, it is important that teachers using traditional textbooks are aware of the hidden messages contained in them so that they may address such issues in their teaching.

Background

The analysis of the textbooks presentations is based on a number of ideas about mathematics and its teaching and learning which are briefly discussed here. Skemp (1976) introduced the terms "relational understanding" and "instrumental understanding" in a discussion of mathematics learning. Relational understanding describes the state of knowing in which the learner builds up an interconnected, structured knowledge. Instrumental understanding describes the type of knowledge in which the learner acquires a set of fixed procedures and rules which exist more or less as individual elements of knowledge to be applied to very specific situations. An alternative formulation of this idea was provided by Hiebert and Lefevre (1986) in their discussion of "conceptual" and "procedural" knowledge in mathematics. They defined conceptual knowledge in similar terms to those of Skemp, that is, as a "connected web of knowledge" (p. 3) and requiring the learner to recognise "its relationship to other pieces of information" (p. 4). Hiebert and Lefevre described procedural knowledge in terms of two kinds of information: knowledge of the language and symbols and the conventions for their use; and knowledge of the rules and procedures used for solving mathematical problems.

Bell (1993) describes "mathematical activity" in terms of a cycle of "mathematization, manipulation, and interpretation" (p. 6). The three stages involve the recognition of a mathematical relationship and its symbolic expression, the manipulation of the symbols to uncover some new aspect, and the interpretation of the result in terms of the situation. Bell noted that traditional mathematics teaching spends most time on the manipulation stage of the cycle.

Teachers' instructional practices are closely related to their views of mathematics and its teaching and learning. Thompson (1992) found support for this relationship, but also showed that the relationships among a teacher's views, beliefs and instructional preferences were highly complex. A teacher's view of mathematics and its teaching and learning may be described as lying somewhere along a continuum. At one extreme, the view of mathematics has been variously described as formalist (van Dormolen, 1985) or instrumentalist (Skemp, 1976; Ernest, 1989). Those holding this view of mathematics see it as a system of facts, rules, theorems and skills in which intuition plays no real part. The associated view of teaching and learning is of a transmission process or a conveyance-container model (Mason, 1989). This model characterises mathematical knowledge and skills as existing entities which have to be acquired by the learner. The emphasis is on the learning of technical terms and their definitions, skills and techniques with much symbolic recording and manipulation.

At the opposite end of the continuum is the view of mathematics as an open, dynamic field which is constantly changing and expanding, an "activist" view (van Dormolen, 1985). Doing mathematics is a process of exploration using activities such as classifying, formalising and quantifying. Intuition is an important element. Teaching entails facilitating the use of these processes by the students. It is possible for an individual teacher to adopt a view of school mathematics and its teaching somewhere between the two ends of the continuum. Cobb (1988) described a continuum of teaching approaches from negotiation (active construction) to imposition (transmission) and noted that most mathematics teaching tended towards the imposition pole.

After more than six years of formal education, secondary school students have formed their own conceptions of what it means to study mathematics in school. This will have a considerable influence on their acceptance of the experiences provided by the teacher. Cobb (1986) provided the example of "students who have constructed instrumental beliefs about mathematics (Skemp, 1976) anticipate that future mathematical experiences will 'fit' these beliefs" (p. 4). A teacher attempting to present mathematics in a more "activist" way may experience resistance from such students. Garofalo (1989) discussed the common beliefs about learning mathematics held by secondary school students in the United States. These include:

> Almost all mathematics problems can be solved by the direct application of the facts, rules, formulas, and procedures shown by the teacher or given in the textbook. Only the mathematics to be tested is important and worth knowing.
>
> Mathematics is created only by very prodigious and creative people; other people just try to learn what is handed down. (Garofalo, 1989, pp.502-503)

In a study of Australian students, Herrington (1990) found similar beliefs and stated that "it is apparent that many students see practising the same question over and over again, or copying notes from the blackboard, as the best way to learn mathematics" (p. 15).

Mathematics textbook analysis

A number of authors have reported on methods of examining various aspects of mathematics textbooks. Fauvel (1991) analysed three early textbooks (from the years 1543, 1743 and 1910) from the viewpoint of what he called the *tone* of the texts, that is, "how the author has worked to influence the way that readers must respond to the book to benefit in the way intended" (p. 111). He described textbook writers as "teachers at a distance" (p. 116) and examined the pedagogic techniques used in the texts. Morgan (1996) employed a linguistic approach to mathematical texts. While her work has focussed mainly on texts written by students, the approach is equally applicable to published textbook material. Using the work of Halliday (1973) she described three aspects of text, namely: the *ideational* function which addresses the nature of mathematics and mathematical activity including the role of people in its creation; the *interpersonal* function which considers the roles and relationships of the reader and author; and the *textual* function which considers the way the text is constructed. Morgan's discussion elaborated the difficulties students appeared to find in producing written mathematical texts in extended assessment tasks to match the expectations of their teachers.

Van Dormolen (1985) provided a method of analysing the content in mathematics textbooks. Terminology and categories used by van Dormolen were adopted and extended in a coding scheme developed by this author in earlier studies (e.g., Shield, 1996) with some minor renaming of categories. A brief overview of the coding scheme textual categories and descriptors is provided. These provide a language for the discussion of the characteristics of mathematical texts.

Elaborations: *kernel* **(K)** - general expressions that have to be learnt - rules, definitions and procedures; *exemplar* **[E]** - a worked example of the procedure or an example of the concept, which is presented with at least one of the following components: *symbolic representation* **[S]** - numerical and/or algebraic symbols: *verbal description* **[V]** - the specific worked example given in words: *diagram* **[D]** - a line drawing to support the exemplar; *table of data* [T] - a collection of data of some kind arranged systematically: *convention* [C] - a verbal statement of a generally accepted practice; *graph* [G] - a Cartesian line graph, bar graph, histogram, etc; *goal statement* [G] - an identification of the concept or procedure that is the subject of the text, often a heading; *justification* [J] - an attempt to show something to be correct by referring to known principles; *link to prior knowledge* [L] - a description of a prior mathematical skill needed for the new procedure, or a description of an everyday example of the concept or procedure; *practice exercises* [P] - a set of questions for the reader, modelled on the exemplar.

All verbal and symbolic statements in the text, including kernels, are described in terms of their "aspect of mathematics" and "level of language" as described below.
Aspects of mathematics: *theoretical* [T] - theorems, definitions, axioms - parts of the structure of the topic; *algorithmic* [A] - explicit methods or "how to do" a specific operation or procedure; *logical* [L] - statements about the way one should work using the theory - an extension of the theory; *methodological* [M] - "how to do" rules but of a more general, heuristic nature than algorithmic - the rule needs interpretation to provide an answer; *conventional* [C]- conventions, how to name a diagram, write a proof.
Levels of language: *particular* - language used in discussing a specific example or case. *generalised* - language conveying the general meaning of a concept or procedure and not related to specific examples. Within each level of language, the statements may be either *procedural* or *descriptive*: *procedural* - step by step instructions; *descriptive* - stating the meaning of an idea or the appearance of an object or diagram. There are therefore four language descriptions: *particular procedural* **[PP]**; *particular descriptive* **[PD]**; *generalised procedural* **[GP]**; *generalised descriptive*.

A text analysed

To illustrate the analysis being adopted in the present study, a section of text from a published textbook by this author (Shield & Wallace, 1988) has been used. The chosen textbook has many similarities to other published mathematics textbooks which are being analysed in the project. While the intention is not to be critical of the writers of successful (in terms of sales) textbooks, it is inevitable that criticisms will be inferred from the findings of the analysis. However the constructive intent of the criticisms is discussed in the conclusion of this paper. Two pages from the textbook are reproduced in Appendix 1 with codes for elaborations (underlined), aspect of mathematics (boxed) and level of language (circled). The two pages were chosen as typical of pages devoted to the exposition of new material. The pages immediately preceding and following those illustrated are discussed later.

While this text contains a little more verbal material than some others, it is still rather "dense" in terms of the number of ideas that have to be remembered by the reader. On the two pages in the appendix there are 4 statements classified as *kernels*, being *generalised* statements of definitions and procedures, for example: "A common factor of two numbers is a factor which occurs in both numbers." The aspects of mathematics being developed are *theoretical* and *methodological* in the kernels, followed by the *algorithmic* use of the mathematics in the worked examples. Each kernel is illustrated with at least one specific example (*exemplar*). There is no use of *justification* in the presentation and unusual cases are not considered. The worked examples of the two main skills being developed are followed by sets of practice exercises. The exercises can be completed by the reader by using the worked examples as models.

The approach to the skills is purely instrumental and at this stage there is no indication of possible uses for the skills being developed. On the previous pages in the textbook, the reader is reminded of what should be prior knowledge of factors, products and prime and composite numbers. There is an investigation using the ideas of the "sieve of Eratosthenes" which talks about Eratosthenes being a Greek mathematician of the third century BC and a friend of Archimedes. The remainder of the chapter after the pages reproduced in Appendix 1 is devoted to division, rules for divisibility, and approximate calculations. There is no further reference to highest common factor.

Interpretation of the text

The main aim of the preliminary phase of the project was to identify the underlying messages which the textbooks convey to teachers and students. The above type of analysis provided the basis for reflection on the views of mathematics and its teaching and learning inherent in the text. It was also necessary to consider the context in which such a text would be used to inform further analysis. Generally the set textbook is the only resource in the students' possession and, as mentioned earlier, is fundamental in the planning of many teachers. While the context, in terms of the views of the teacher and the range of experienced offered in the classroom, may vary considerably, the mathematics textbooks being analysed appear to have been written with a "tell-show-do" approach to mathematics in the minds of the authors. The repeated use of kernels, worked examples and exercises in the example text analysed here, fits this pattern.

The use of kernels expressing theoretical and methodological aspects of mathematics in generalised descriptive and procedural language conveys the idea that mathematics consists of complete and established principles which have to be acquired by the learner. There is no "mathematical activity" (Bell, 1993, p. 6) with its cycle of "mathematization, manipulation and interpretation" evident here. The ideas are presented in complete form and illustrated abstractly. While there has been some attempt to develop the ideas from prior knowledge of factors, the end result is an "instrumentalist/formalist" view of mathematics presented in a transmission mode. The lack of any reason for doing the procedure and any real use of the procedure reinforces this view. The presentation shows little evidence of the development of conceptual knowledge and concentrates on the two kinds of procedural knowledge identified by Hiebert and Lefevre (1986), that is knowledge of the appropriate language and symbols and knowledge of the rules and procedures as demonstrated in the kernels and exemplars. If this presentation were to be followed closely by the teacher, there would be little space for the active construction of the ideas by the learners.

The sample text may be characterised using Halliday's (1973) aspects of text. The *ideational* function addresses the nature of mathematics and mathematical activity which is essentially instrumentalist. It also includes the role of people in the construction of the mathematics. While the sample textbook and all the others so far analysed include references to people in brief historical snippets, there is no real feeling that the mathematics being learnt is a human endeavour. Nominalised mathematical objects are manipulated logically. The *interpersonal* function considers the roles and relationships of the reader and author. The sample textbook presentation has an authoritative tenor in which the authors are transmitting the necessary knowledge to the reader with the reader expected to just accept what is being presented. While the sample presentation, and most others analysed, include statements of the type "we can see that . . . ", the use of the inclusive "we" does not really have the function of including the reader in the development of the idea. There is little else that the reader can do but accept what is being presented and practice the procedures using the exercises provided. It would be interesting to know how mathematics learners interpret such inclusive statements, or if they have any impact at all. Finally, the *textual*function considers how the text is constructed. As discussed earlier, the sample text consists mainly of sequences of kernels, exemplars and practice exercises typical of a tell-show-do approach. Generally this approach is likely to match the expectations of the teachers and students using the text.

Conclusion

In this paper, a method of analysing the content of mathematics textbooks has been outlined. The mathematical teaching function of each statement in a text can be defined as a kernel, exemplar, link or other categories. Identification of the aspect of mathematics being expressed and the level of language being used provides a characterisation of each mathematical statement. From this analysis, the underlying messages about the nature of mathematics and its teaching and learning that the text might convey to teachers and students may be inferred. While the influence of the textbook on mathematics classrooms is known to be strong, it is not possible to know the extent to which the underlying messages influence classroom proceedings in individual cases.

The textbooks analysed so far are generally similar to the sample text demonstrated in this paper. Clearly, the textbooks themselves do not convey the intent of recent reports and syllabuses, even though they were written in response to these documents. While it may not be possible, in a printed form, to totally meet the needs of new syllabuses, it should be possible to develop textbook presentations which come much closer than at present. It has been shown that many mathematics teachers at all levels rely heavily on the textbooks for their curricular decisions and while some may be able to recognise and overcome the limitations of the textbooks, many cannot. There is also the question of the role of the learners in using the textbooks and their expectations about what doing mathematics in school entails. The project aims to further such questions.

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APPENDIX 1

Factorisation

To factorise a number means to write it as a product of two or more factors, e.g. $12 = 4 \times 3$, $24 = 2 \times 3 \times 4$.

Any composite number can be factorised. A prime number such as 23 can be written as 1×23, but in mathematics this serves little purpose. Usually, numbers are written as a product of all prime factors, e.g.

$36 = 3 \times 12$ (3 prime, 12 composite)
$= 3 \times 3 \times 4$ (3 prime, 4 composite)
$= 3 \times 3 \times 2 \times 2$ (all prime)

A factor tree is a way of organising this.

Example

Write (a) 36 and (b) 84 as products of prime factors.

Solutions

(a)
36
3 × 12
3 × 3 × 4
3 × 3 × 2 × 2

$36 = 3 \times 3 \times 2 \times 2$

(b)
84
7 × 12
7 × 3 × 4
7 × 3 × 2 × 2

$84 = 7 \times 3 \times 2 \times 2$

Note: It does not matter which factors you start with, e.g. for (b) above:

84
4 × 21
2 × 2 × 3 × 7

$84 = 2 \times 2 \times 3 \times 7$

SHARPEN YOUR SKILLS 3

A. Complete these factor trees.

1. 18
9 × 2

2. 48
4 × 12

3. 240
24 × 10

B. Write each of these numbers as a product of prime factors using a factor tree.

1. 30
2. 42
3. 27
4. 210
5. 240
6. 98
7. 216
8. 315
9. 960
10. 968

HIGHEST COMMON FACTOR

A common factor of two numbers is a factor which occurs in both numbers. (For this definition, name the: *item*, *class*, and *feature*.)

For example,

$24 = 2 \times 2 \times 2 \times 3$ (prime factors)
$36 = 2 \times 2 \times 3 \times 3$

From this we can see that there are some common factors of 24 and 36—two 2s and one 3.

The highest common factor (HCF) of two numbers is the biggest number which is a factor of both numbers. (Definition: *item? class? feature?*)

The HCF can be found by finding all the prime factors which are common to both numbers.

Example

Find the highest common factor (HCF) of (a) 24 and 36 and (b) 48 and 18.

Solutions

(a) $24 = 2 \times 2 \times 2 \times 3$
$36 = 2 \times 2 \times 3 \times 3$

The common factors are 2, 2 and 3.

$\text{HCF} = 2 \times 2 \times 3$
$= 12$

(b) $48 = 2 \times 2 \times 2 \times 2 \times 3$
$18 = 2 \times 3 \times 3$
$\text{HCF} = 2 \times 3$
$= 6$

48
4 × 12
2 × 2 × 3 × 4
2 × 2 × 3 × 2 × 2

Note: It is best to write the prime factorisations of the two numbers with the prime factors in increasing order of size.

SHARPEN YOUR SKILLS 4

Find the highest common factor (HCF) for each of these pairs of numbers.

1. 8 and 20
2. 24 and 40
3. 42 and 14
4. 90 and 60
5. 24 and 48
6. 36 and 16
7. 78 and 26
8. 50 and 20
9. 108 and 72
10. 27 and 63
11. 15 and 18
12. 24 and 40
13. 36 and 60
14. 45 and 75
15. 28 and 42

LOWEST COMMON MULTIPLE

The lowest common multiple (LCM) of two numbers is the smallest number which is a multiple of both numbers. (Definition: *item? class? feature?*)